Research Article

Journal of Innovative Research in Teacher Education, 4(2), 167-183
ISSN: 2757-6116
http://www.jirte.org
DOI: 10.29329/jirte.2023.572.1
Received: 17/02/2023
Revised: 24/05/2023
Accepted: 04/06/2023

How pre-service teachers teach fraction comparisons in mathematics pedagogy*

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Abstract

The present study aimed to investigate pre-service teachers' mathematics teaching when responding to virtual pupils' unexpected mathematical questions concerning how to sort fractions. The research was conducted with 102 pre-service teachers participating in a teacher education program specializing in upper elementary school mathematics. The main data collection strategy took the form of video recordings of teaching sessions involving semi-virtual simulations. The recordings were analyzed through a qualitative explorative analysis process involving three phases, focusing on the object of learning and how the pre-service teachers handled the mathematics content pertaining to comparing fractions. The simulated teaching activity, together with the large sample, allowed us to discover patterns concerning how PSTs teach when they are asked unexpected questions related to fractions. The results show qualitatively different ways of teaching. PSTs draw pupils' attention to different learning objects by using different representations and more or less correct mathematics. This study offers teacher educators knowledge about how PSTs teach when confronted by students with unexpected questions concerning fractions and can therefore support teacher educators in interpreting students' mathematics instruction and help them decide how best to support PSTs' teaching instruction.

Keywords: Pre-service teacher, Fraction teaching, Simulation, Variation theory.

Cite: Sveider, C., Samuelsson, J., Thorsten, A. & Samuelsson, M. (2023). How pre-service teachers teach fraction comparisons in mathematics pedagogy. *Journal of Innovative Research in Teacher Education, 4*(2), 167-183. https://doi.org/10.29329/jirte.2023.572.1

*This research was approved by The Swedish Ethical Review Authority.

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INTRODUCTION

Mathematics is a gatekeeper to academic and professional success, which gives the subject a special position in the curriculum. Investing in teacher training programs should therefore pay off when it comes to pupils' learning of mathematics. Providing teacher educators with awareness of how best to support pre-service teachers' development of a knowledge base concerning teaching mathematics in a professional way is an important research topic (Blömeke & Kaiser, 2017; Blömeke et al., 2022; Tatto et al., 2008).

Over the past two decades, researchers have established models of teacher competence that conceptualize a broad range of cognitive and affective competencies that teachers must accomplish (Blömeke & Kaiser, 2017). Earlier on, these models pulled from Shulman's 1986 work, which argued that teachers' domain-specific knowledge could be allocated into, in our case, mathematical content knowledge (MCK), and pedagogical content knowledge (MPCK). Empirical studies have shown that these components constitute two knowledge dimensions (e.g., Baumert et al., 2010; Blömeke et al., 2016). MCK comprises knowledge about the math that teachers must teach at different levels, and MPCK covers curricular knowledge and planning for teaching mathematics. Concerning the weak correlation between teachers' knowledge and pupils' learning. Blömeke et al. (2022) proposed a more elaborate model with a mediating process between teacher knowledge and pupil achievement. They argue that this process is a domain-specific cognitive skill; a skill of how best to perceive, interpret, and make decisions in classroom settings. Their results show that teachers´ cognitive skills (PID; Perception, Interpretation, Decision-making) in a classroom situation have a direct effect on pupils' achievement, while MCK and MPCK have an indirect effect on pupils' mathematical achievement. When investigating pre-service teachers (PSTs), PID seems to be important in helping educators perceive, interpret, and decide upon what they should teach in teacher education programs. Tatto et al. (2008) argue for several core situations (planning, teaching, and evaluating) mathematics teachers are expected to manage when they teach mathematics. To manage these situations, teachers need different types of knowledge. They discuss mathematical curricular knowledge, knowledge of planning mathematics teaching and learning and enacting mathematics teaching. Several researchers have investigated PSTs' mathematical knowledge (MCK) and their mathematical pedagogical knowledge (MPCK) to understand PSTs' challenges concerning these areas (Blömeke et al., 2022). However, to date, there are few studies on how PSTs enact mathematics teaching when they respond to unexpected issues related to specific content. To fully understand the challenges PSTs face in the process of becoming a mathematics teacher, their interactions with pupils merit investigation (Tatto et al., 2008).

Tatto et al. (2008) claim that teachers employing mathematics for teaching and learning need to: (i) analyze or evaluate pupils' mathematical solutions or arguments; (ii) analyze the content of pupils' questions; (iii) diagnose typical pupil responses, including misconceptions; (iv) explain or represent mathematical concepts or procedures; (v) generate fruitful questions; (vi) respond to unexpected mathematical issues; and (vii) provide appropriate feedback. In our study, we investigate how PSTs respond to unexpected mathematical questions concerning how to compare fractions. To answer unexpected questions PSTs must analyze the pupil's question and interpret the pupil's conception (or misconception) of the mathematical principle at hand before they decide how they will teach (Blömeke et al., 2022).

Besides or beyond traditional modes of investigating PST's teaching, today's teacher educators can also use different kinds of simulations to understand PST's teaching (Badiee & Kaufman, 2015; Kaufman & Ireland, 2016, Sveider, 2021). Semi-virtual simulations, such as TeachLivE, include virtual characters with different personality traits, who are handled by a human (Dieker et al., 2014; Ersozlu, et al., 2021). PSTs who practice teaching with semi-virtual simulations provide the research community with unique opportunities to study *how* PSTs enact teaching when they perceive, interpret, and decide what to do in specific classroom situations. Investigating a specific context provides evidence that can be used to further define the general or main idea of a topic and prove that it is valid in other contexts. The specific context investigated in this study is PSTs' teaching of fractions. Fractions were chosen since they are one of the most important areas in mathematics education (Pedersen & Bjerre, 2021) It's also one of the areas in mathematics that is most complicated and challenging for pupils to learn (Barbieri et al. 2020), and many pupils struggle to learn fractions throughout their school years (Fazio, DeWolf, & Siegler, 2016; Schneider & Siegler, 2010).

Strategies Comparing Fractions

Several studies have focused on pupils' strategies for finding solutions when comparing fractions, such as (i) the GAP thinking strategy, (ii) the residual thinking strategy, (iii) the reference point strategy, (iv) the concrete representation strategy, and (v) the comparing denominators strategy (e.g., Pearn & Stephens, 2004; Yang & Lai, 2013).

Several studies have shown that some pupils use an informal solution strategy comparing two fractions; namely, a strategy referred to as the *gap thinking strategy* (e.g., Post & Cramer, 1987; Pearn & Stephens, 2004; Mitchell & Horne, 2010). Pupils using the gap thinking strategy argue that $\frac{3}{5} > \frac{5}{8}$ because "there is less gap"; i.e., a difference between 3 and 5 in the fraction expression $\frac{3}{5}$ and between 5 and 8 in the fraction expression $\frac{5}{8}$ (Pearn & Stephens, 2004). One consequence of such a strategy is that pupils believe that the fractional expressions $\frac{3}{4}$ and $\frac{2}{3}$ have the same value because the 'gaps' between the numerator and the denominator in both fractional expressions are equal (Mitchell & Horne, 2010). Mitchell and Horne (2010) identified two additional variants of the gap thinking strategy in comparing fractions. One variant of the gap strategy was that pupils argued that the aforementioned fractions were equal in size because the numerators in the two fractions were 'smaller' than the denominator. The second variant meant that pupils thought that both fractions were equal, since both fractions needed the addition of one part to become a whole (p. 415). The studies show that regardless of the variant of this strategy, it is not a useful way to compare fractions. Clarke and Roche (2009) show that pupils use two sustainable informal solution strategies to compare fractions. These two solution strategies are referred to as *the residual thinking strategy* and *the reference point strategy*. The solution strategy with *residual thinking* refers to the amount needed for the fraction expression to become ahole. For example, if the fraction expressions are $\frac{5}{6}$ and $\frac{7}{8}$, pupils think that the fraction expression $\frac{5}{6}$ requires $\frac{1}{6}$ to become whole, whereas $\frac{7}{8}$ needs $\frac{1}{8}$ to become whole. Since $\frac{1}{8}$ is less than $\frac{1}{6}$, the fraction expression $\frac{7}{8}$ is the larger of the two. *The reference point strategy* means that pupil uses a reference point—usually $\frac{1}{2}$ but also one—when they compare fractions (Clarke & Roche, 2009). To determine which of the fractions $\frac{5}{8}$ and $\frac{3}{7}$ is the largest, a reference point of $\frac{1}{2}$ is used. Here it is seen that $\frac{5}{8}$ is greater than $\frac{1}{2}$, while $\frac{3}{7}$ is less than $\frac{1}{2}$; therefore $\frac{5}{8}$ is greater than $\frac{3}{7}$.

Mack (1990) shows that pupils could correctly compare two fractions by using an informal, *concrete representation strategy;* such as equal circles divided into different parts depending on the size of the denominator. In the explanation of which fraction is greater, the pupils referred to pictures of the circles. Furthermore, the results show that the pupils related their thinking to everyday life. This was shown when the pupils in a teaching situation could correctly determine which pizza slice was the largest. Mack (1990) also shows that pupils who could detect what fraction expression was largest when they use concrete representations failed to resize the fractional expressions $\frac{1}{8}$ and $\frac{1}{6}$ at the same lesson. Several pupils argued that $\frac{1}{8}$ was greater than $\frac{1}{6}$. The pupils explained this by saying that 8 is greater than 6. This, according to Mack (1990), is an indication that the pupils in the study did not use their previous knowledge of comparing fractions, even though the same fractional data were used. One interpretation of the result is that the step between the informal solution strategy and the formal solution strategy was too large for the pupils in the study (Yang & Lai, 2013).

Why Fractions are so hard to Understand?

There are several reasons why pupils may have problems understanding numbers in fractional form. One reason is how fractions should be interpreted. Numbers in fractional form are expressed by the form a/b, where a and b belong to the numerator and the denominator, and are expressed as numbers. Many pupils have several years of experience with numbers before they are taught about fractions. Pupils' previous experiences of integers as independent numbers may overshadow the understanding that numbers in fractional form should be understood as a relationship between the numbers 'a' and 'b' (DeWolf & Vosniadou, 2015; Siegler & Lortie-Forgues, 2017). This is referred to as *the whole number bias* (Ni & Zhou, 2005; Siegler et al., 2011), and may, for example, mean that pupils perceive the fraction 1/4 to be greater than the fraction $\frac{1}{3}$, because 4 is greater than 3 (Barbieri et.al., 2020; Lortie-Forgues et al., 2015; Siegler & Lortie-Forgues, 2017). Additional reasons why pupils may have problems understanding fractions are that fractions have multifaceted meanings, such as equal division, scale, ratio, and proportion (Cramer et al., 2002; Kilpatrick et al., 2001; McMullen et al., 2014; Resnick et al., 2016). Pupils' knowledge of numbers in fractional form can therefore be divided into different aspects, so-called 'subgroups': (i) sub-whole, (ii) ratio, (iii) operator, (iv) the result of a division, and (v) measurement in which the sub-sub-whole plays a superior role (Kieren, 1980). According to Kieren (1980), pupils need to know of each of the fractional subgroups, and of how each of the subgroups are related to each other.

Theoretical Framework

This study is about how pre-service teachers (PSTs) handle pupils' mathematical issues when they teach fractions. To handle pupils' issues, PSTs need to perceive, interpret, and decide (PID) how to explain and represent the mathematics principle in question and how best to handle the lesson's content. The handling of content regards choosing what content is presented during instruction and which aspects of said content are emphasized during instruction (Marton & Booth, 1997). To refer to both these issues, this project is influenced by the variation theory presented by Marton (2015). In variation theory, researchers focus on the object of learning. Marton (2015) divides the object of learning into three different types: the intended object of learning, the enacted object of learning, and the lived object of learning. In our study, we focus on the *enacted* object of learning and analyze what learning opportunities are offered during the lesson (Marton & Booth, 1997).

Variation theory has been used in our study to analyze how PSTs diagnose pupils' issues and handle their issues concerning resizing fractions. Marton (2015) asserts that some aspects of an object of learning are necessary to discern the object of learning in a targeted way. If these aspects are not yet discerned, teachers need to make them discernable for pupils. To do this, teachers can use variation as a tool. According to Marton and Pang (2013), "meanings are acquired from experiencing differences against a background of sameness, rather than experiencing sameness against a background of differences" (p.24). Based on this assumption, pupils' learning opportunities will increase if teachers vary the aspect to be discerned while allowing other aspects to remain *invariant*; that is, constant, thus creating *contrast* (Marton, 2015).

There are several ways teachers can use contrasts. Mathematics teachers often use different types of representations, such as everyday incidents, manipulatable materials, pictures, and oral and written language (e.g., Rau & Matthews, 2017). "Manipulatable materials" refers either to materials that are constructed for educational purposes—e.g., geoboards or cubes—or real-life materials—e.g., pipes or toothpicks (Szendrei, 1996). Sveider (2021) demonstrates that teachers use representations in qualitatively different ways concerning which representation they use and how the different representations are linked. When several representations are linked to each other, a system of representations is created, which enables pupils to discern the object of learning (Goldin & Shteingold, 2001). Teachers make the links between representations in various ways, such as linking together representations used during the lesson, as well as linking them with representations used previously. When teachers link representations simultaneously, they use verbal and non-verbal actions. Sveider (2021) also argues that verbal actions are used when teachers describe and explain similarities between representations. When teachers use non-verbal actions, they point at or circle essential aspects of said representations. Using multiple representations and linking them together is a mark of quality instruction in mathematics (Schlesinger et al., 2018).

Rationale for the Study

In the present study, the aim was to investigate PSTs' mathematics teaching when responding to virtual pupils' unexpected questions regarding how to sort fractions. There are, to date, few studies on how PSTs enact teaching when they respond to unexpected issues related to specific content. Earlier research mainly focuses on PSTs' MCK and MPCK (Blömeke et al., 2022; Tatto et al., 2008); which is why the results of this study are important to fully understand PSTs' challenges when it comes to teaching fractions. The results will therefore help teacher educators draw attention to important aspects of how to enact mathematics teaching concerning fractions, but also how to perceive, interpret, and decide upon feedback when they supervise their teacher students in classrooms. Thus, our results will therefore pay off concerning pupils' learning of mathematics (Tatto et al., 2008; Blömeke & Kaiser, 2017; Blömeke et al., 2022).

METHOD

The main data collection strategy involved video recordings of the teaching sessions. We argue that this is essential, given that classrooms are such complex environments (Cochran-Smith, 2015; Doyle, 2006). Video recordings gave the researchers time to observe PSTs while they were teaching in the classroom, and to analyze what took place. Therefore, video recordings offer us a powerful tool to analyze the interaction concerning what content is brought to the foreground in the instruction and how representations were used.

Participants and Context of the Study

The present study used convenience sampling (Bryman, 2018) and included 102 PSTs enrolled in teacher education specializing in upper elementary schools in Sweden. One group of PSTs (41) took part in a course about mathematic didactics in 2019, and the other group of PSTs (61), took part in the same course in 2020. All PSTs in both cohorts were informed that a specific part of the mathematic didactics course included teaching virtual pupils in a simulation. They also received information that the simulation training with the virtual pupils was part of a research project in which they were invited to participate. All PSTs agreed to be recorded when they taught the virtual pupils. To handle confidentiality, no names or years were included in the paper. The recordings have only been analyzed by the researchers and have been stored in portable hard drives (Swedish Research Council, 2017).

Setting

The simulation training was done using the semi-virtual simulation TeachLivE in a small room that had been rearranged so that the PSTs had a desk for their lesson plan, materials in front of them, and a whiteboard behind them. In front of their desk were virtual pupils, shown on a screen. Communication to and from the virtual pupils was done with the help of a conference sound system. All PSTs came to the training session in groups of three. They had prepared a lesson plan for three lessons in advance. Each PST taught once and observed two lessons. All groups visited the simulation three times. As there were three PSTs in the room, each of whom knew what the others were supposed to be teaching, support was available during the lessons from peers and instructors, as needed. All PSTs met five virtual pupils, each about 10 years of age (Ava, Dev, Ethan, Jasmine, and Savannah), and each virtual student had a different 'personality'. The PSTs were forced to handle (a) the GAP thinking strategy, (b) the residual thinking strategy, (c) the reference point strategy, (d) the concrete representation strategy, and (e) the comparing denominator strategy. The session ended with reflections and feedback from all participants, led by the two instructors. One instructor was specialized in mathematics teaching, while the other was specialized in leadership.

Analytical Process

The qualitative explorative analysis process started as the researchers, one by one, examined randomly chosen recordings among the 102 total recordings. The analysis was based on an a priori expectation (Malinowski, 1922) that something was interesting in the variation in how PSTs enacted mathematics teaching when responding to virtual pupils' unexpected mathematical issues concerning how to sort fractions. The first phase of watching recordings was followed by a second collective phase when the researchers came together and looked at the consistent recordings. Doing so also included reasoning and arguments about what had been seen in the recordings. In this phase, several ideas were discussed before the research team decided to focus on *(a) the enacted object of learning, and (b) the handling of representations,* since these aspects are central when teaching mathematics. In the third phase, the research team analyzed how the PSTs taught the mathematical content of the lesson plan, focusing on the *knowledge possible to discern* and the *representations used*. In this process, variation theory was used to see whether and how PSTs used contrasts to make the content discernible to the pupils. The analysis resulted in several qualitatively different ways of teaching each aspect of the course content, based on an inductive analysis. The analysis of teaching $\frac{1}{3} < resulted$ed in variation regarding whether the PSTs focused *on rules or conceptual understandings*. The analysis of how PSTs taught about denominators and numerators instead concerned variation of mathematical correctness in teaching, since this was the most interesting thing that varied. The results were summarized in a model. The model was tested against the data and revised several times (c.f. Lakoff, 1987; Maxwell, 1992). To ensure validity, we used a range of strategies, including member checking, peer debriefing, and reflexivity (Bryman, 2018). These strategies helped us minimize potential biases and increased the trustworthiness of the research findings.

Ethical Considerations

This research was ethically approved by The Swedish Ethical Review Authority.

FINDINGS

The results will be presented in two sections concerning how PSTs respond to unexpected mathematical issues when teaching how to compare fractions with a denominator of 1 ($\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$). In the first section we focus on (a) how PSTs respond to the misconception $\frac{1}{3} < \frac{1}{4}$. In the second section, we focus on (b) how PSTs respond to what functions the denominator and the numerator have in a fraction.

How PSTs respond to Misconceptions Regarding Comparisons of $\frac{1}{3}$ and $\frac{1}{4}$

In this first section, we illustrate how virtual pupils demonstrated misconceptions about how to compare the fraction expression ($\frac{1}{4} > \frac{1}{3}$). PSTs began to teach virtual pupils how to compare fractions. Seven qualitatively different strategies were used to answer the unexpected question "Why is $\frac{1}{3}$ bigger than $\frac{1}{4}$, when 4 is more than 3?" These seven categories were clustered into two different themes: (a) a focus on rules, explained using mathematical representations; or (b) a focus on conceptual understanding, explained using concrete representations. Table 2 summarizes our results regarding how PSTs respond to virtual pupils expressing the misconception that $\frac{1}{3} > \frac{1}{4}$, concerning teaching focus, knowledge possible to discern, and representations.

Table 1. Overview of Teaching Focus PSTs used to Help Pupils Understand that $\frac{1}{3} > \frac{1}{4}$

Teaching focus	Knowledge possible to discern	Representations used by PSTs
Focus on rules	The rule that: $\frac{1}{3} > \frac{1}{4}$	Connections are made between mathematical symbols, and written and spoken instruction.
	Rule: the bigger dominator, the smaller the fraction. Not a sufficient condition.	Connections are made between mathematical symbols, and written and spoken instruction.
	Rule: the magnitude of the numerator and denominator affect the size of the fraction.	Connections are made between mathematical symbols, and written and spoken instruction.
Focus on conceptual understanding	A conceptual understanding of why $\frac{1}{3} > \frac{1}{4}$	Connections are made between manipulatable materials, mathematical symbols, and written and spoken instruction.
	A conceptual understanding of why $\frac{1}{3} > \frac{1}{4}$	No connections are made between representations.
	A conceptual understanding of why $\frac{1}{3} > \frac{1}{4}$	No connections are made between representations, which include carelessly drawn pictures, mathematical symbols, and written and spoken instruction.
	A conceptual understanding that fractions can be seen as a division	Connections are made between real- life material, mathematical symbols, and written and spoken instruction.

Focus on Rules Explaining with Mathematical Representations

In the theme, a focus on rules explaining with mathematical representations, we found three different strategies (see Table 2) PSTs used in responding to unexpected questions regarding how to compare fractions. All three strategies focus on learning mathematical rules with respect to comparing fractions with the support of mathematical symbols.

In our first example, the PST handed over the explanation to another virtual pupil. That pupil presents how to resize the fraction expression using language connected to a mathematical context.

Ava	I think that a quarter must be the largest. Because four is more than both two and Three. A quarter is the largest, and then comes a third and last one-second.
PST	Ehh okay, do we have any other suggestions.
Savannah	One half is the largest, then a third and then comes a quarter.
PST	Good, one half is the largest, and then a third, and the smallest is a quarter.

The pupil declares that $\frac{1}{3}$ is bigger than $\frac{1}{4}$. No explanations about fundamental mathematical ideas regarding the unit fraction are presented by the virtual pupil or the PST. The opportunity to discern how to compare fractions is limited.

In our second example, the PST refers to a rule that is not entirely mathematically correct, since the PST only describes the relationship between the denominator and forgets to address how the numerator affects the size of the fraction.

PST	Which fraction is the largest; a third or a quarter?
Ethan	That is easy, I can count: 1, 2, 3, 4. 4 is bigger, a quarter is the largest.
PST	You could think that, but as you say 4 is bigger than 3, but not when it is about fractions. The larger the denominator is, or as you say, the number down below the fraction line, the smaller the fraction.

Saying that when the denominator gets larger, the fraction gets smaller, the PST focuses on one fundamental aspect when comparing fractions, which is the bigger the dominator, the smaller the fraction. Accordingly, the PST draws a verbal contrast between different fraction units, and it is therefore possible to discern that when comparing unit fractions, the bigger the denominator, the smaller the fraction. The explanation is connected to a mathematical context when the PST uses a verbal representation to reinforce the rule regarding how to compare fractions with respect to the denominator. But since the explanation only focuses on the function of the denominator, the knowledge available to the pupils regarding how to compare fractions is insufficient.

In our third example, the PSTs draw attention to rules related to how the magnitude of the numerator and denominator affect the size of the fraction. The PST says, while simultaneously pointing to the denominator, that the bigger the denominator is, the smaller the fraction—if the numerators are equal.

PST	Now I have written two fractions on the board [writes a $\frac{1}{3}$ and $\frac{1}{4}$], which is the largest?
Savannah	The one with the four, four is bigger than 3.
PST	Ok, like this, when it comes to fraction, or fraction form, you must look at the digit here and here [points at the numerator and denominator]. The bigger the denominator is the smaller the number is...if the numbers have the same numerator.

In this example, the numerator is kept invariant, while the denominator is varied when PSTs compare the fractions. The PST is making a contrast between the fractions. In this way, pupils can discern the relative size of the fractions, which here means that the larger the denominator, the smaller the number, if the numerators of the fractions are equal.

Thus, PSTs use qualitatively different strategies with respect to what knowledge becomes visible, and the representations used. In the above examples, they draw attention to rules explaining why $\frac{1}{3}$ is bigger than $\frac{1}{4}$. The PSTs give the pupils sufficient conditions to fully master comparing different fractions.

A focus on Conceptual Understanding, Explained Using Concrete Representations

In the theme, which is focused on conceptual understanding explained using concrete representations, we found four different strategies adopted by PSTs when they answered pupils' unexpected questions regarding fraction comparisons. All four strategies focused on learning mathematical concepts by comparing fractions, and were supported by concrete representations.

In our first example, the PSTs use manipulatable materials—namely, pre-produced rectangles—to illustrate differences between $\frac{1}{3}$ and $\frac{1}{4}$. By contrasting different parts of the rectangles $\frac{1}{3}$ and $\frac{1}{4}$, the PSTs prove that the bigger the denominator, the smaller the fraction.

PST	We have two different answers [about which is greater a $\frac{1}{4}$ or a $\frac{1}{3}$]. Now this is going to be exciting to see who is right. I thought I'd show you, with fraction rectangles. In addition, as we have on the board now, we have a whole, this green is a whole [shows a green elongated rectangle]. Now we are going to see how many quarters fits in one whole [pastes parts below] and the blue ones are quarters. Now we'll see that four quarters make a whole, but if we were to look at thirds [pastes thirds below). Now you see that, these whites are thirds, three thirds make a whole. Which of the pieces is largest?
Dev	Yes, it was like I said that a third is bigger than a quarter.
PST	Yes, that is true, even if four is greater than three, a third is greater than a quarter.

With the support of connections between multiple representations—concrete material and mathematical concepts—the pupils are given the opportunity to discern which fraction is the largest. The PSTs use 'the part of a whole' strategy, and contrast how many parts fit in one rectangle (divided into thirds and fourths). This can help pupils discern that the bigger the denominator, the smaller the fraction. In our second example, the PSTs draw pictures to convince the pupils what fraction is the biggest. The example illustrates the importance of drawing the invariant precisely so as to be able to contrast fractions with different denominators.

PST	Which is greater, $\frac{1}{3}$ or $\frac{1}{4}$ [The numbers are written on the board]. Does anyone see or do we need to paint this. Which is bigger, $\frac{1}{3}$ or $\frac{1}{4}$?
Savannah	Eeeeh, 4 is bigger than 3, so a $\frac{1}{4}$ must be bigger than a $\frac{1}{3}$. I think so anyway, I'm not sure.
PST	I can understand it looks a bit like that, because 4 is bigger than 3. However, if we draw it, you'll see. Here we have a pizza, [draws a circle on the board], then we divide it into four.
Ethan	I love pizza, it is so good!
PST	Then I fill in $\frac{1}{4}$. Then we make another pizza, which we divide, into three parts and then we fill in one part. Now maybe it's a little easier to see which one is the biggest? Is there anyone who can look at the pictures? Savannah
Savannah	Yes, now it is very clear. I thought four was more than three.
PST	It's easy to think so, that's why it's so good to draw the pizzas or some other types of symbols, we will not go through that today, but pizzas are great tricks if you are unsure. Because it is easy to do when numbers are different sizes to take the largest number. Here it is the opposite.

By using pictures, the PST draws a contrast between the denominators in the two fractions, making it possible for the pupils to discern which fraction is the largest. However, it could be difficult for pupils to understand what the pictures represent, because the PST makes no connection between the pictures and the fractions.

In the third example, the PSTs mean to convince the pupils that $\frac{1}{3}$ is larger than $\frac{1}{4}$ with the support of drawn rectangles. To help pupils decide which fraction is the largest, the PST draws two rectangles. Since the rectangles are carelessly drawn (the rectangles are different sizes), the pictures will not help pupils decide which fraction is the largest. As in the example above, the PSTs do not make any connections between the representations. This leads to the contrast between the denominators being overshadowed.

PST	Is it now so that you can see with the naked eye which fraction is the largest, and which is the smallest? I'll wait a bit so you can think. J would you like to try to explain.
Jasmine	I´m not sure, D knows for sure.
PST	You can guess.
Jasmine	Did you say I could just look?

PST	Yes, if we look at these two numbers [points at the rectangles $\frac{1}{3}$ and $\frac{1}{4}$]. Which one looks the biggest?
Jasmine	I think the quarter looks a little bigger than the third, that is, the part that is filled in or they are the same size.
PST	Great that you are trying to talk about how you think. Is there anyone who thinks otherwise?
Dev	I think if you look at how they are drawn, I suppose it's meant to, it is difficult to draw exactly, I think you've tried to draw them the same size.

In the fourth example in this theme, the PST listens to the pupils' thoughts about how the task should be solved. The PST is tolerant of pupils' errors, and when a pupil offers an incorrect explanation of which fraction is the largest, the PST tries to convince the pupil of their error with the help of an example taken from the pupil's everyday life.

PST	What do you think about this? [Points at $\frac{1}{3} < \frac{1}{4}$]. Is this correct?
Savannah	I think so
PST	What did you say? You think so?
Savannah	That it is true as you have written
PST	You think that´s right
Savannah	Yes, because, 4 is very much, so, I do like this, if you´re going to compare like this, I think that you take 4–1 is 3 and 3–1 is 2 and 3 is more than 2, so that, then it must be correct
PST	OK, interesting thought Savannah. Does anyone else have any other suggestions or do you agree with Savannah?
Ava	I disagree, I do not understand how you thought Savannah.
PST	It can be a bit complicated. This is 1, up here [point to the numerator in $\frac{1}{8}$], I understand if you want to mix it in Savannah, but one might think that these are parts, as you said Ava, if you are four people and are going to share a roll cake [point to the 4], or if you are three people and are going to share a roll cake [point to the 3] which are the same size, you get more if you are only three people. So, what I wrote on the board is false; it should actually say $\frac{1}{4} < \frac{1}{3}$.

Our interpretation is that the explanations is in some way suitable for the pupils, since the explanation is connected to real-life representations of something the pupils will recognize. But in the example, the PST uses the numerator when she argues that there are smaller pieces if four people share the cake than if three people do it. The PST draws attention to the division process of one cake, rather than showing that the numerator shows how many parts are in play. The language used by the PST is imprecise. For example, the PST does not specify that it is the denominator that varies.

Therefore, the results illustrate qualitatively different ways of addressing pupils' misconceptions about fractions, focusing on conceptual understanding. In the above examples, PSTs use multiple representations to explain why $\frac{1}{3}$ is bigger than $\frac{1}{4}$. The PST's drawing skills and how they connect different representations affect their instruction; and through their instruction, pupils' opportunities to discern what they are supposed to discern.

How PSTs respond to Questions about the Functions of the Numerator and the Denominator

When PSTs teach fraction comparisons, pupils have questions about the function of numerators and denominators. We have found three qualitatively different ways PSTs teach about the function of the numerator and the denominator. PSTs' different ways of enacting mathematics teaching affect their explanations, and thereby pupils' opportunities to discern the learning object.

Table 2. Overview of Teaching Foci PSTs used to Help Pupils Understand the Function of the Numerator and Denominator

Teaching	Knowledge possible to discern	Representations used by PSTs
With no formal mistakes	The concept of the numerator and the denominator	Connections are made between mathematical symbols, written and spoken instruction, everyday life representations, and pictures.
With formal mistakes regarding the concept of fractions	The concept of fractions and the concept of division are equal	Connections are made between mathematical symbols, and written and spoken instruction.
With formal mistakes regarding the numerator	That the numerator means one whole instead of one part	Connections are made between mathematical symbols, written and spoken instruction, and everyday life representations

Teaching with no Formal Mistakes

In our first example, the PSTs does not make any content-related or formal mistakes when teaching about the function of the denominator and the numerator. The PST uses the pupils' misunderstanding of what the denominator and numerator mean as an opportunity to teach about and illustrate their functions. The PST draws two circles and splits them into two halves.

Figure 1. Two Circles Drawn on The Whiteboard Representing one whole

Savannah I don´t really get it, is it because the two that stand there, are one for the blue [refers to the blue. Circle representing one whole] and one for the yellow pizza [refers to the yellow circle representing two halves]. Because it was two pizzas, you said, so when you write two-thirds down there, is it to say that we have two pizzas.

PST No that is not a correct, number two here, [points to the denominator in $\frac{2}{2}$] means how many parts might fit in the pizza, one whole [takes the two yellow fraction circles and puts them over the blue circle]. The numerator up here means there are two parts right now. However, if I remove this one [removes one yellow fraction part] then it becomes one half.]

With the help of multiple representations and relevant examples, the functions of the numerators and the denominators are explained and contrasted. As a result, pupils are given the opportunity to discern between the functions of the numerator and the denominator.

Teaching with Formal Mistakes with Respect to the Concept of Fractions

In our second example, the PSTs does not separate the difference between a rational number and the fact that a division is an operation between two integers. The PST makes the pupils aware that the vinculum separating a numerator and a denominator does not mean the same thing if it is a fraction line, or a division line. On the other hand, pupils are not made aware of the difference between a fraction, which represent a rational number, and division, which is an operation between two integers.

Savannah	I have one question; actually, I have two questions. In third grade, we worked on division, and then you have the same signs as in fractions. Is it the same thing or is it something entirely different now that we are working with fraction?
PST	When we write is as a fraction then it is called a fraction line [points to the fraction line]. When we talk about division for example one divided by two, then we have a division sign. And then, it doesn't mean the same thing.
Savannah	So, they are two completely different things
PST	Yes, exactly
Savannah	It has nothing to do with each other, you could say, it is just that it happens to look the same.

The example contains a potential contrast between fractions and division as an operation between two integers, but the PST does not make the pupils aware of this difference Therefore, pupils are afforded scant opportunity to discern the differences between these lines. Instead, this tactic can even reinforce pupils' misconceptions about the two lines, and lead them to believe that the concept of fractions and the concept of division are equal.

Teaching with Formal Mistakes Regarding the Function of the Numerator

The third example shows how the PST teaches about the function of a numerator and a denominator, but they make mistakes concerning the function of the numerator in a fraction. The PST is tolerant of pupils´ errors regarding the numerator´s function. But instead of correcting the pupil's misconceptions, the PST follows and emphasizes the pupils´ misconceptions; namely, that the numerator means how many of the whole the fraction represents.

Savannah	Just a question so I know what the number stands for. You said before when you showed a fourth, you pointed to the four and said it was the number of persons who shared a pizza and then you pointed at the number one and said it meant one pizza.
PST	Exactly
Savannah	Is that how you should think or?
PST	Yes, this was just one example, and you can always get help by thinking of a pizza.
Savannah	So, the one means how many pizzas you share, and the third how many persons are sharing.
PST	Yes, in this example. One pizza and three persons.
Savannah	I get it. If it should stand two thirds, then you have two pizzas that three persons will share.
PST	Yes, very good!
Dev	I´m wondering if that´s true.
PST	Like I said in this example it is true; it is one pizza and three who share

In the example above, we can see the PST argue that in the fraction $\frac{1}{3}$, the numerator means one whole instead of how many thirds. This leads to the pupils being given an incorrect explanation, which makes it difficult for them to understand and discern the numerator's function in a fraction.

Therefore, PSTs use qualitatively different ways of teaching about the function of numerators and denominators. PSTs' conceptions or misconceptions of fractions seem to affect how they teach fractions and, accordingly, which learning objects the pupils are able to discern.

DISCUSSION AND CONCLUSION

The results of this study offer teacher educators knowledge about how PSTs teach when confronted with unexpected questions from students with respect to fractions. The results can support teacher educators when they interpret students' mathematics instruction to decide how best to support the student's teaching ability.

Blömeke and Kaiser (2017) argues that being able to answer unexpected questions is a central competence that you must accomplish as a mathematics teacher. We chose to study how PSTs employ mathematics when they teach fractions for two reasons: a) several studies point out that fractions are hard to understand, and b) fraction knowledge predicts more advanced mathematical performance (Siegler & Lortie-Forgues, 2017). Earlier research show that teachers' ability to perceive, interpret, and decide (PID) actions in the classroom affects pupils' performance in mathematics (Blömeke et al., 2022). In our study, we focus on what happened after PID during a mathematics lesson. The result therefore extends our knowledge of PSTs' challenges in teaching mathematics, since earlier research mainly focused on PSTs' mathematical knowledge and mathematical pedagogical knowledge. These challenges can be seen when PSTs try to make the object of learning discernable to pupils (Marton, 2015).

The program "TeachLivE" offered the research team a unique opportunity to present unexpected questions with specific content to students. This allowed us to employ a large sample. Therefore, the simulated teaching activity, together with the large sample, gave us the opportunity to discover patterns with respect to how PSTs teach when they are asked unexpected questions related to fractions. The results show qualitatively different ways of teaching. PSTs draw pupils' attention to different learning objects by using different representations and correct or incorrect mathematics. The result also showed how PSTs' lack of MCK with respect to fractions affected the answers they gave in response to unexpected questions. It was evident that some PSTs didn't understand that fractions have multifaceted meanings; such as equal division, scale, ratio, and proportion (Cramer et al., 2002; Kilpatrick et al., 2001; McMullen et al., 2014; Resnick et al., 2016). To be able to answer unexpected questions with respect to fractions, PSTs need to be aware of all the relevant subgroups (Kieren, 1980).

The PSTs brought different learning objects to the foreground when teaching pupils why $\frac{1}{3} > \frac{1}{4}$. Some PSTs focused on rules and did not try to offer an explanation that pupils could understand. Others tried to help pupils discern the learning object and understand which expression was the largest. The enacted object of learning therefore offers pupils different learning opportunities (Marton, 2015). Depending on how PSTs enact the learning object, our results show that students are given different teaching strategies to understand how to compare fractions—from how to perform specific procedures and apply mathematical rules without necessarily understanding why they are working, to understanding the underlying principles and making connections between mathematical ideas. In some cases, pupils are even taught how to compare fractions based on an incorrect teaching strategy.

Usually, the examples that the PSTs chose contained an inherent contrast, but sometimes the contrast is not made visible and the students themselves must render the learning object, which can be limiting. Our conclusion is that although it is important that the examples themselves carry a contrast, it is also important that teachers draw pupils' direct attention to essential aspects of the learning object.

When answering unexpected questions from pupils, PSTs use different strategies when comparing fractions (e.g., Pearn & Stephens, 2004; Yang & Lai, 2013). Our study shows that PSTs often use the concrete representation strategy and the comparing denominators strategy. When they use concrete representations, they contrast different parts of a whole, such as $\frac{1}{4}$ (Marton, 2015). When using the denominator strategy, they contrast numbers with respect to the number line. If this more informal strategy is used correctly, pupils' understanding of ordered fractions can be reinforced (Mack, 1990). However, we can see that the PSTs sometimes applied the strategy in an imprecise way, which can hinder pupils' learning.

Therefore, even though the PSTs in this study were at an equal level in respect to their education, they showed qualitatively different ways of answering students' questions and enacting mathematics teaching. Differences are found in several aspects of the PSTs' teaching when they answered unexpected questions related to how to compare fractions. Our study shows that they differed in their teaching strategy, the knowledge possible for students to discern, the representations they use to make contrasts, and their mathematical correctness when teaching. Even though they have similar educational backgrounds, and are roughly the same age, they teach in qualitatively different ways. There are probably several explanations as to why this is the case, but one interpretation is the knowledge of mathematics teaching that PSTs have acquired/obtained during their teacher training is not sufficiently able, or does not have the correct focus, to be able to challenge PSTs' previous perceptions of mathematics and mathematics teaching.

This study, like all qualitative studies, has limitations regarding its generalizability. Three important issues need further investigation to strengthen the generalizability of the study. First, researchers need to continue to investigate PSTs' ways of handling specific content concerning pupils' unexpected questions. Secondly, researchers need to investigate PSTs' behavior with real pupils to see whether there are any similarities in different contexts. Third, researchers also need to investigate PSTs' teaching concerning other content in mathematics to see whether there are similarities when teaching different learning objectives.

Statement of Researchers

Researchers' contribution rate statement: All authors equally contributed to the study.
Conflict statement: The corresponding author declares that she has no conflict of interest.
Support and thanks: None

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